

Lab 5

Fourier series in

“Experiment” and “Theory”.

1.1 Motivation:

In this lab, we will investigate the Fourier series of some common waveforms using both the symbolic calculation features in Matlab and Simulink. In preparation for the lab you should read it over paying particular attention to the material in section 1.2.

1.2 Introduction and background theory:

By using complex Fourier series, we can represent any periodic signal $G(t)$ with period $T = 2\pi/\omega$ as a sum of constant coefficients C_k multiplied by quadrature functions (sinusoidal functions).

$$g(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega t} \quad (1)$$

Where C_k = complex Fourier series coefficients.

And $e^{jk\omega t}$ = the phasor components of the periodic signal ($\cos(k\omega t) + j \sin(k\omega t)$).

The values of C_k (CFS coefficients) are given by the following equation:

$$C_k = \frac{1}{T} \int_0^T g(t) e^{-jk\omega t} dt \quad (2)$$

Where the integration is carried out over one period of the periodic waveform. Physically the magnitude of the complex values of the C_k are easily seen from equation (1) to correspond to the amplitude of the component at each frequency. The square magnitude of the coefficients corresponds to the power at each frequency. The phase of the complex numbers, C_k determines the phase of the sinusoid that corresponds to the frequency component.

Intuitively the integral of the product of the function and the complex sinusoid is a measure of the amount of ‘overlap’ of the sinusoid and the function. If $g(t)$ were in fact a sinusoid at ω then the overlap would be perfect and the appropriate C_k would be the only non zero term. C_0 is the measure of the overlap with the constant value 1 and is clearly the average or DC value of the signal over one period.

Alternately if the signal $g(t)$ is a real function then the real transform can be used. In this representation the signal can be expressed as:

$$g(t) = \sum_{k=-\infty}^{\infty} \operatorname{Re}[C_k] \cos(2\pi kt/T) - \operatorname{Im}[C_k] \sin(2\pi kt/T) \quad (3)$$

Where

$$\operatorname{Re}[C_k] = \frac{2}{T} \int_0^T g(t) \cos(k\omega t/T) dt \quad (4)$$

And

$$\operatorname{Im}[C_k] = -\frac{2}{T} \int_0^T g(t) \sin(k\omega t/T) dt \quad (5)$$

Note that if $g(t)$ is a symmetric function then all of the $\operatorname{Im}[C_k]$ will be zero and similarly if $g(t)$ is antisymmetric then all of the $\operatorname{Re}[C_k]$ will be zero. The practical consequence of this is that if you can shift the time origin to give a purely symmetric or antisymmetric function then you only have to do half the number of integrals to find the coefficients. The relative magnitudes of the real and imaginary parts determines the phase of that frequency contribution to the signal (just like phasors in EE101).

In the limit that the signal has an infinite period the coefficients become a continuous function of frequency (rather than just having discrete values) and the coefficients of the fourier series becomes the Fourier Transform (which you be learning more about in lecture and future labs):

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt \quad (6)$$

Where $\omega = 2\pi f$.

1.3 Square wave Analysis (breaking it up into sine waves).

Using Matlab construct a vector corresponding to a rectangular periodic signal (“square wave”, Use the Matlab function “square(t)”) whose amplitude changes between 0 and 1 volt, the width τ of each rectangular pulse is 4 sec. and the period is $2\tau = T_0 = 8$ sec. Set the phase of the square wave so that it is centered to be +1 from -2 to $+2$ etc., so the period $T_0 = 2\tau = 8$ units of time, switching sign at times $= 2 \pm 4n$ for integers n . Note then the square wave is symmetric about 0. What does this immediately tell us about the Real and Imaginary parts of C_k ? The time average of this signal is $\frac{1}{2}$, because half of the time it is +1 and half of the time it is 0. Therefore the first coefficient of the Fourier series is $C_0 = \frac{1}{2}$.

Then plot this signal using MATLAB.

If this square wave is translated downwards by $\frac{1}{2}$, the result has $C_0 = 0$ because half the time the result is $+\frac{1}{2}$ and the other half the time it is $-\frac{1}{2}$. This would form a pure AC signal with no DC component expressed in language you may be more familiar with. We shall not translate it downwards in this example.

By centering the square wave so it is symmetric about 0, the Fourier series is a series in cosines only of frequency $n2\pi f = n\pi/\tau$ radians. The symmetry assures that no sine $n2\pi f$ terms appear in the Fourier series which will simplify the next part of this exercise.

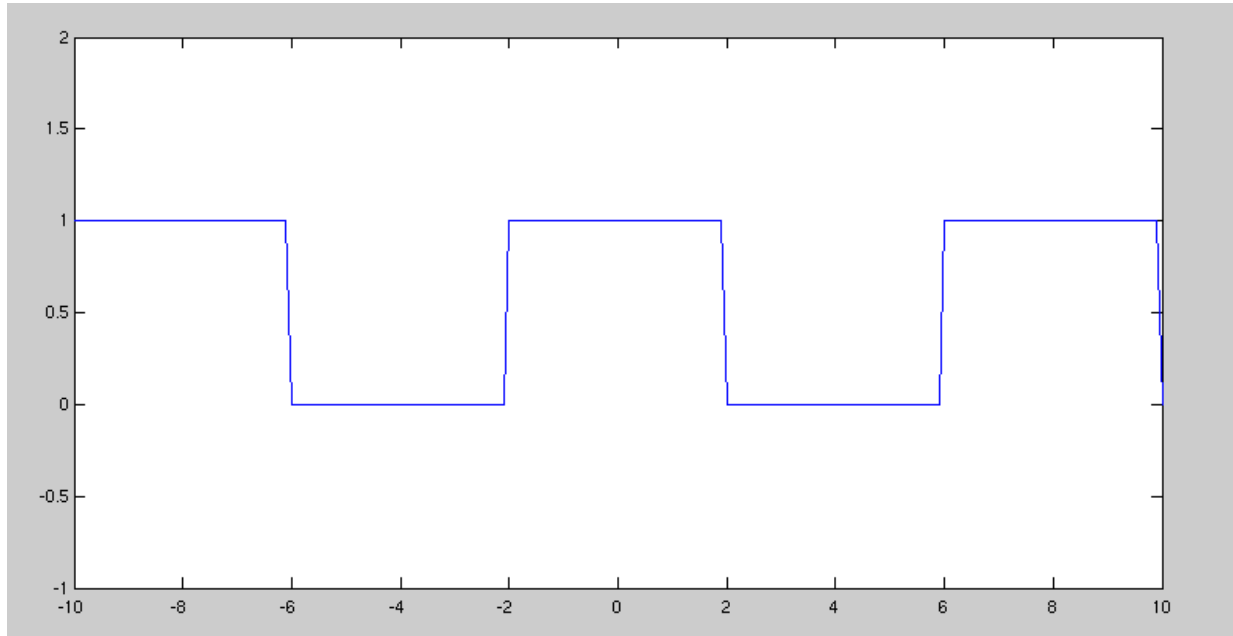


Fig. 1: A square wave

Now we will use the Fourier *transform* to find the values of C_k for the Fourier *series*. The reasons for using this mathematical trick here is that we can evaluate the integral in equation (2) easily, this illustrates the deep connection between the Fourier Transform and the Fourier Series and we get to use the sinc function which is extensively used so you may as well get more familiar with it.

Later we will get the Fourier series coefficients using the more usual integral expressions shown above.

We start with $g(t)$ as the square wave as generated above.

The Fourier transform of a single period of this waveform, an isolated pulse is:

$$G(f) = A\tau \operatorname{sinc}(f\tau)$$

The sinc function is defined as $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$

Now the really interesting thing is that the Fourier series coefficients are given by the values of the Fourier transform evaluated at the frequencies of the harmonics of the phasors that sum to make the series. Mathematically speaking if f_0 is the fundamental frequency of the periodic signal then the harmonics that make up the Fourier series are at kf_0 , where k is an integer. To get the Fourier *Series* coefficients one then evaluates the Fourier *Transform* (in this case $G(f)$ above) at these discrete frequencies.

Figure 2 below shows a graph of the sinc function (the Fourier Transform of a single pulse) and dots representing its value at a discrete set of evenly spaced frequencies. These discrete values are the values of the C_k of the Fourier series of a pulse with a periodicity equal to the spacing in frequency between the dots. If one imagines the single pulse repeating less and less frequently then the dots get closer and closer together. In the limit that it never repeats the dots all smooch together to make a continuous line: the Fourier transform. This is very cool. This show that the Fourier Transform results comes from the Fourier series in the limit that the periodic waveform never repeats.

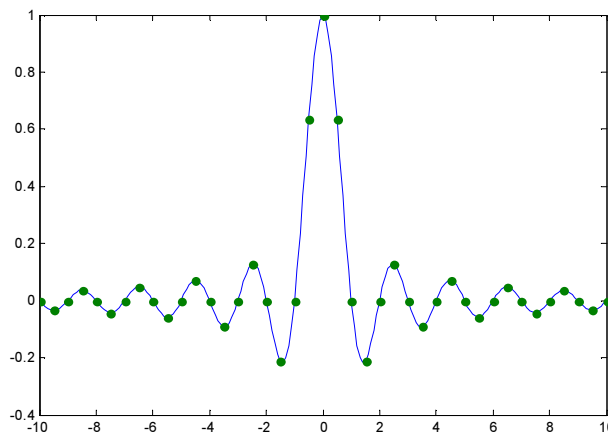


Figure 2, A sinc function and its values at discrete points evenly spaced in frequency shown by the dots.

To evaluate the Fourier Transform at the discrete frequencies for the case of our square wave substitute $kf_0 = f$ into the Fourier Transform:

$$G(kf_0) = A\tau \text{sinc}(kf_0\tau)$$

The Fourier series coefficients are given by:

$$C_k = \frac{1}{T_0} G(kf_0) = \frac{1}{8} \cdot 1 \cdot 4 \cdot \text{sinc}(k/2) = \frac{1}{2} \text{sinc}(k/2) = \frac{1}{2} \frac{\sin(\frac{\pi}{2}k)}{\frac{\pi}{2}k} = \frac{1}{\pi k} \sin(\frac{\pi}{2}k)$$

So $C_0 = 1/2$, $C_1 = 1/\pi$, $C_2 = 0$, $C_3 = -1/3\pi$, $C_4 = 0$, $C_5 = 1/5\pi$...

and decreases as $1/k$ with alternating coefficients = 0 and with alternating signs.
To implement the summation, put the corresponding value of C_k in the expressions:

$$g(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk2\pi ft} = \dots C_{-2} e^{-j4\pi ft} + C_{-1} e^{-j2\pi ft} + C_0 + C_1 e^{j2\pi ft} + C_2 e^{j4\pi ft} \dots$$

Which can be rewritten as:

$$g(t) = C_0 + 2\{C_1 \cos(2\pi ft) + C_3 \cos(2\pi 3ft) + \dots\}$$

Let $n = k/2$ so summing over n gives

$$\begin{aligned} g(t) &= C_0 + 2 \sum_{n=1}^{\infty} C_n \cos(2\pi nft) \\ &= 1/2 + (2/\pi) \cos(t\pi/4) - (2/3\pi) \cos(3\pi t/4) + \dots \end{aligned}$$

We will now use this result to synthesize a square wave in Matlab.

1.4 Squarewave Synthesis Simulation in Simulink

Open a new model in Simulink. Put 4 sine wave generators, 3 adders, and 5 scopes in the model. Connect 4 scopes one to each sine wave generator to make sure you are getting the correct frequency and amplitude sine wave from each sine wave generator.

Set the frequency of the first sine generator to 1/8 Hz the fundamental of the square wave you generated above (careful you must convert to radians per second to enter the correct radians in the frequency of the sine generator) with an amplitude of as given above. The others will need to be set to the appropriate harmonics of this fundamental, as given in the formulas above, start with the amplitude of each of these set to 0.

Using the “simulation/configurationparameters” menu to set the “max step size” to 0.01 to be sure that you have enough points on each waveform.

One at a time set the amplitude of each harmonic to the correct value to generate a square wave, sum the outputs adjusted to the correct frequency and amplitude to correspond to the first terms in the Fourier series of the square wave, and record the resulting partial sums off the remaining scope.

What happens to the slope of the waveform at the zero crossings as each new harmonic is added? Qualitatively what does this result imply regarding the bandwidth of a signal and its Risetime? This is an observation of critical concern for digital system design.

What happens to the flat parts at the top and bottom of the waveform as more of the harmonics are added? Is the sum approaching a better squarewave as more of the harmonics are added?

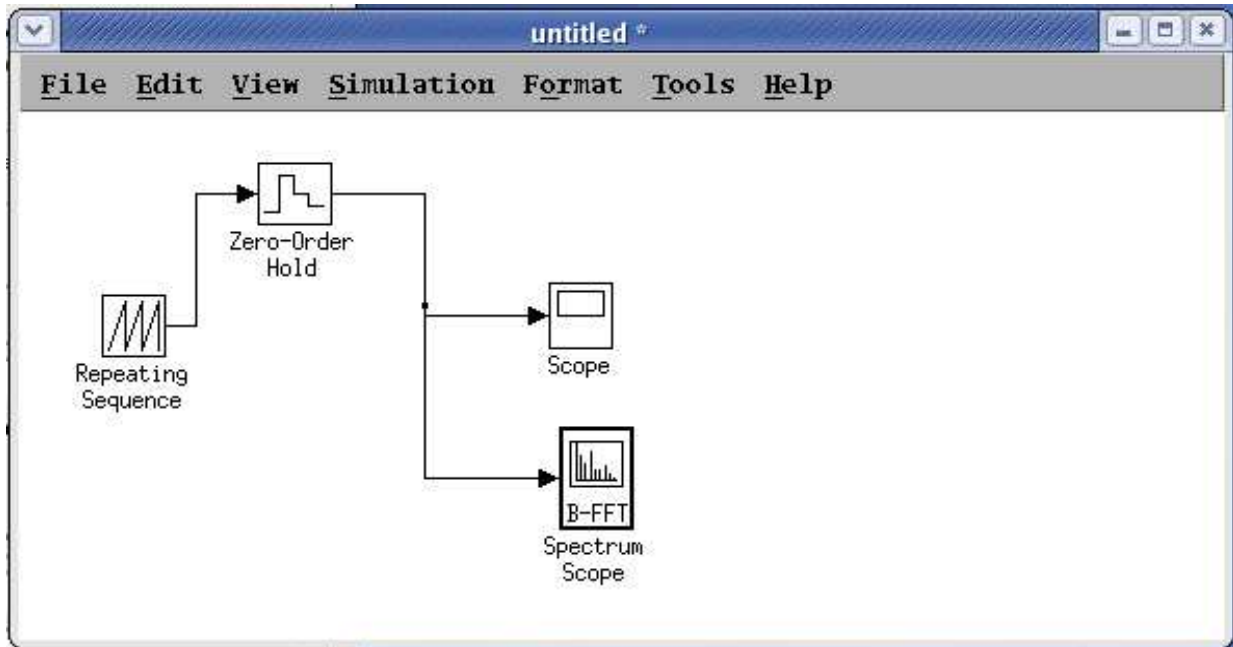
All of these sine waves are being added together with a particular phase relationship to each other.

Does the phase matter?

To find out, shift the phase of the second sine wave generator by half of its period and record the waveform. Does this phase shift affect the shape of the waveform? What does this phase shift do to the Power in each harmonic? If you were looking at this waveform on the spectrum analyzer (which displays the power spectral density) would you expect the display to change with this phase shift?

1.5 The sawtooth wave simulated

Open up a new Simulink model and set it up to generate a sampled sawtooth wave that is fed to a scope and a spectrum analyzer as shown below:



Set the time values of the repeating sequence block to $[0 \ 1]$, this generates a waveform that repeats once every second (1 Hz). Set the output values of the repeating sequence to $[-1 \ 1]$ to generate a waveform that changes from -1 to 1 over this one second interval. Set the sampling interval of the Zero-Order hold to 0.01 so that we have enough points to get good high frequency resolution. How many points per period does this give us?

Set the simulation/configuration parameters menu items to give a Stop time of 50 sec for good low frequency resolution and set the Solver Option type to “Fixed Step” so that the Spectrum scope block doesn’t complain. How many total points does this give us in the simulation?

In the Spectrum Scope block choose the spectrum units as dBW, the spectrum type to “one sided” (so this will act approximately like the real spectrum analyzer in the lab) and check the Buffer input check box. For the best looking spectra results you want the Spectrum scope to use as many points as possible so set the buffer size to be the largest power of

two that is less than your total number of points. This block is actually calculating the power spectrum using an algorithm called an FFT (Fast Fourier Transform) and this algorithm needs to use a number of points that is equal to a power of two. The Buffer size setting tells the algorithm how many points to use.

Look at and record the spectrum analyzer output from 0 to 10 Hz. Record the values of the power at 1, 2 and 3 Hz. You can zoom in to get these values more accurately.

What should the theoretical value at zero Hz be for this waveform? Why? What is the meaning of the value of the spectrum at zero Hz? What is the actual value that you see in the simulation? Why do you think it is not equal to its theoretical value?

Change the output values of the repeating sequence block to [0 2]. How does this change the oscilloscope waveform? How does this change the spectrum? Now theoretically what should the value of the power spectrum at zero Hz be? (Note: you won't actually see this exact value because we are not actually using an infinite waveform with infinite frequency resolution, just as with a real waveform and a real spectrum analyzer in the lab.) Record the values of the peaks at the fundamental and the next two harmonics for this wave form. How have they changed (approximately)? Why do you think that this is so?

Mess around with the sawtooth and see how changes in amplitude, phase, frequency etc. affect the spectrum. Try to predict qualitatively what you expect to see with each change, see what happens and then refine your intuition about how the time and frequency spectra are related.

1.6 The saw tooth wave in theory

We can now use the symbolic capabilities in Matlab to calculate the theoretical values of the coefficients of the Fourier series for the waveform used in the last example using Matlab.

Derive a symbolic expression that fits the sawtooth waveform above over the interval from 0 to 1 sec. It won't match the actual sawtooth outside this interval but that won't matter because to get the coefficients we only need to integrate this waveform between 0 and 1.

Using the syms statement define v and t as symbolic variables.

Create an expression for C_0 (the Dc coefficient of the Fourier series) by integrating this function using the symbolic int function to integrate this waveform between 0 and 1. Record its value.

Similarly compute the coefficients for C_1, C_2 and C_3 by integrating the product of this waveform and sinusoids at 1 Hz, 2 Hz and 3 Hz with this function over the interval from 0 to 1 (over one full period). This is just the implementation of the formula for the fourier series coefficient.

$$\text{Re}[C_k] = \frac{2}{T} \int_0^T g(t) \cos(k\omega t / T) dt$$

$$\text{Im}[C_k] = -\frac{2}{T} \int_0^T g(t) \sin(k\omega t / T) dt$$

The physical interpretation of each coefficient can be thought of as a measure of how well the waveform “overlaps” each sine wave.

Compare the ratios of c_2/c_1 and c_3/c_1 you get here to those you measured for this waveform above. (Careful: these values are in

amplitude and the spectrum scope gives the answers in power in dB you will need to convert). Are the ratios close?

Near as I can tell the absolute values given by the version of the spectrum scope that I am running are not correct. How do yours compare?

Lab Report Worksheet

1.3 Square wave Analysis (breaking it up into sine waves).

1) What does the function symmetry immediately tell us about the Real and Imaginary parts of C_k ?

2) Provide a plot the square wave signal using MATLAB.

1.4 Squarewave Synthesis Simulation in Simulink

1) Partial sums for first 4 terms

2) What happens to the slope of the waveform at the zero crossings as each new harmonic is added?

3) Qualitatively what does this result imply regarding the bandwidth of a signal and its Risetime?

4) What happens to the flat parts at the top and bottom of the waveform as more of the harmonics are added?

5) Is the sum approaching a better squarewave as more of the harmonics are added?

6) Does this phase shift affect the shape of the waveform? Show plot.

7) What does this phase shift do to the Power in each harmonic?

8) If you were looking at this waveform on the spectrum analyzer (which displays the power spectral density) would you expect the display to change with this phase shift?

1.5 The sawtooth wave simulated

How many points per period does this give us?

How many total points does this give us in the simulation?

Values of the power at 1,2 and 3 Hz.

What should the theoretical value at zero Hz be for this waveform?

Why?

What is the meaning of the value of the spectrum at zero Hz?

What is the actual value that you see in the simulation?

Why do you think it is not equal to its theoretical value?

How does changing the output values to $[0,2]$ this change the oscilloscope waveform?

How does this change the spectrum?

Now theoretically what should the value of the power spectrum at zero Hz be?

Record the values of the peaks at the fundamental and the next two harmonics for this wave form.

How have they changed (approximately)?

Why do you think that this is so?

1.6 The saw tooth wave in theory

Coefficients C_1 , C_2 and C_3

Compare the ratios of c_2/c_1 and c_3/c_1 for the two methods of getting the coefficients.

Are the ratios close?

Comment on the correspondence of the actual values for the coefficients?